Improved Formulations for the Analysis of Convecting and Radiating Finned Surfaces

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The study of the heat flow within finned surfaces involving combined convective and radiative heat dissipation is conventionally based upon unidirectional analyses with attention restricted solely to the fin side. In this study the heat conduction within the interface to which the fins are attached, and the heat transfer from the unfinned side of this interface are also considered. The general situation in which the fins and the base surface have different thermal conductivities and different surface emissivities is examined. Fin-to-base-surface, fin-to-environment, fin-to-fin, and base-surface-to-environment radiant interactions are all accounted for, but the convective fluids are assumed to be radiatively transparent. One- and two-dimensional formulations are developed for analyzing the heat flow. These represent a major extension of the previously employed formulations. The results obtained indicate, in particular, that the previously employed formulations are inadequate for the effective design of finned heat exchangers.

Nomenclature

AUG1 = one-dimensional augmentation factor
AUG2 = two-dimensional augmentation factor
Bi_1 = h_1P/k_w, Biot number
Bi_2 = h_2P/k_w, Biot number
F_{A-B} = view factor from surface A to surface B
g = irradiation, W/m^2
G = $g/\sigma\theta_1^4$, dimensionless irradiation
h = heat transfer coefficient, W/mK
k = thermal conductivity, W/m^{2}K
l = fin length, m
L = l/P
N = number of nodal points
Nc_2 = $\sigma P\theta_1^3/k_w$, dimensionless radiation parameter
P = half fin pitch, m
Q = heat transfer rate, W
t = half fin base thickness, m
T = t/P
w = wall thickness, m
W = w/P
x = longitudinal displacement, m
X = x/P
y = transverse displacement, m
Y = y/P
ϵ = emissivity
κ = k_f/k_w
θ = temperature distribution, K
θ_1 = plain-side fluid temperature, K
θ_{2f},θ_{2e} = fin-side fluid and fin-side environment temperatures, K
φ = θ/θ_1, dimensionless temperature distribution
σ = Stefan-Boltzmann constant, W/m^{2}K^4

Subscripts

1 = plain side
2 = fin side
f = fin
w = wall

Superscripts

$(\ )^*$ = unfinned wall
$(\hat{\ })$ = isothermal base surface

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Introduction

THE theoretical study of the heat flow within finned heat exchangers is of considerable practical importance because of the extensive utilization of finned surfaces for heat transfer enhancement in applications varying from air-cooled heat exchangers in the process industries to heat rejection equipment in space vehicles. The accurate prediction of the thermal performance of finned heat exchangers is essential for compact and efficient design. However, the analysis of such devices is conventionally based upon several simplifying assumptions, in particular, that the heat flow is unidirectional. This assumption facilitates a considerable reduction in the complexity of the analysis. In fact, for problems involving purely convective heat transfer from finned surfaces, the one-dimensional approximation, in general, permits an analytical treatment, e.g., Refs. 1-3.

The applicability of the one-dimensional approximation has been investigated extensively, but only for finned surfaces involving purely convective heat dissipation.[4-9] The early investigations restricted attention solely to the fin side and concluded that two-dimensional effects are negligible provided the transverse Biot number is much less than unity.[4-6] However, recent investigations of the combined fin and supporting surface have shown that the presence of fins induces two-dimensional effects within the supporting surface and these may in turn act to produce two-dimensional variations within the fin.[7-9] Suryanarayana[8] has reported that the difference between the heat transfer rates predicted by the one- and two-dimensional analyses can be as much as 80%. It is therefore essential for the effective design of finned heat exchangers to consider the complete fin assembly and to employ a multidimensional analysis.

The study of the heat flow within finned surfaces involving radiative heat dissipation is particularly relevant to the design of heat exchangers operating at high temperatures or in atmosphere-free environments.[10-15] The analysis of such systems has progressed considerably since the original investigations which examined isolated black fins operating in the absence of any surface convection, e.g., Refs. 10 and 11. The more recent investigations have considered combined convective and radiative heat dissipation from arrays of gray fins which radiatively interact with adjacent fins and also the base surface, e.g., Refs. 14 and 15. However, even these advanced formulations are unsuitable for the effective design of finned heat exchangers because they are based upon unidirectional heat flow analyses with attention restricted solely to the fin side, i.e., they fail to account for the thermal interaction between the fins and the supporting surface.

The primary objective of the present investigation is to develop an accurate formulation for analyzing the heat flow within finned surfaces which dissipate heat by both convection and radiation. First, a one-dimensional formulation is devised which considers the complete fin assembly, i.e., both the fins and the supporting surface simultaneously. The general situation in which the fins and the supporting surface have different thermal conductivities and different emissivities is examined. This formulation in itself represents a major extension of the previous work, e.g., Refs. 10-15, as the previous studies have restricted attention solely to the fin side and have considered only situations in which the fins and the base surface have the same emissivity. However, a further and more fundamental extension is also devised. The general fin assembly situation is analyzed on the basis of two-dimensional heat flow. This represents a considerable increase in the complexity of the analysis. However, it is shown that a recently developed implementation of the boundary integral equation (BIE) method[16] is capable of handling these complexities without difficulty, and gives accurate solutions with relatively modest demands of computational storage and time.

Finally, the validity of the one-dimensional approximation is investigated by comparing the solutions predicted by the one- and two-dimensional analyses.

Analysis

Assumptions

Consider a heat exchanger comprised of equally spaced longitudinal rectangular fins attached to a plane wall, as depicted schematically in Fig. 1. In this study a theoretical representation of this device is developed on the basis of the following assumptions.

1) The wall and fin materials are isotropic and have constant thermal conductivities.

2) There is perfect contact between the wall and the fins.

3) Convective heat exchange is governed by Newton's law of cooling.

4) The surface heat exchange on the unfinned side of the fin assembly is purely convective, while that on the finned side is a combination of convection and radiation.

5) All radiating surfaces are opaque, grey, and diffuse.

6) The radiative environment behaves as an isothermal black surface.

7) The convective fluids are radiatively transparent.

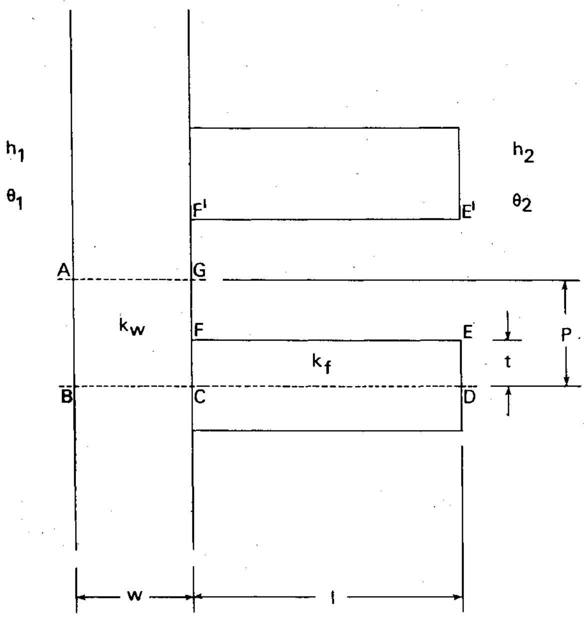

Fig. 1 Schematic representation of a fin assembly.

Assumptions 1-3 and 5-7 are consistent with those conventionally employed in the analysis of convecting and radiating finned surfaces. Assumption 4 indicates that the radiative heat exchange on the unfinned side of the fin assembly is to be neglected. This simplification is introduced primarily in order to facilitate a reduction in the complexity of the analysis, but it is not unrepresentative of the actual physical situation. In practical applications of finned heat exchangers the unfinned side convection is relatively high and, consequently, only at very elevated temperatures is there any appreciable radiative contribution to the total heat transfer.

One-Dimensional Formulation

For steady-state one-dimensional heat conduction, the determination of the fin assembly temperature distribution requires the simultaneous solution of the energy equations:

Within the wall

$$\frac{d^2}{dX^2}\varphi_w(X)=0 \tag{1}$$

and, within the fin

$$\frac{d^2}{dX^2}\varphi_f(X)-\frac{Bi_2}{\kappa T}[\varphi_f(X)-\varphi_{2f}]-\epsilon_f\frac{Nc_2}{\kappa T}\times[\varphi_f^4(X)-G_f(X)]=0 \tag{2}$$

combined with the irradiation equations

$$G_w(0)=\int_{X=0}^{L}F_{\text{FG}-dX}[\epsilon_f\varphi_f^4(X)+(1-\epsilon_f)G_f(X)]+\int_{X'=0}^{L}F_{\text{FG}-dX'}[\epsilon_f\varphi_f^4(X)+(1-\epsilon_f)G_f(X)]+F_{\text{FG}-\text{EE}'}\varphi_{2e}^4 \tag{3}$$

and

$$G_f(X)=\int_{X=0}^{L}F_{dX-\text{FF}'}[\epsilon_w\varphi_w^4(0)+(1-\epsilon_w)G_w(0)]+\int_{X'=0}^{L}F_{dX-dX'}[\epsilon_f\varphi_f^4(X)+(1-\epsilon_f)G_f(X)]+F_{dX-\text{EE}'}\varphi_{2e}^4 \tag{4}$$

subject to the boundary conditions

$$\frac{d}{dX}\varphi_w(X)=-Bi_1[1-\varphi_w(X)] \quad \text{on } X=-W \tag{5a}$$

$$\varphi_w(X)=\varphi_f(X) \quad \text{on } X=0 \tag{5b}$$

$$\frac{d}{dX}\varphi_w(X)=\kappa T\frac{d}{dX}\varphi_f(X)-Bi_2(1-T)[\varphi_w(0)-\varphi_{2f}]-\epsilon_w Nc_2(1-T)[\varphi_w^4(0)-G_w(0)] \quad \text{on } X=0 \tag{5c}$$

$$\frac{d}{dX}\varphi_f(X)=-\frac{Bi_2}{\kappa}[\varphi_f(X)-\varphi_{2f}]-\frac{Nc_2}{\kappa}[\varphi_f^4(X)-\varphi_{2e}^4] \quad \text{on } X=L \tag{5d}$$

The energy Eq. (1) is obtained by performing an energy balance on an infinitesimal element of the wall, while the energy Eq. (2) and the irradiation Eqs. (3) and (4) are obtained by performing an energy balance on an infinitesimal element in an arbitrary fin and combining this with the radiant flux balances between two adjacent fins, the included base surface, and the environment.

The radiant interactions have been modeled in terms of irradiation because this permits a more concise mathematical representation than the conventional radiosity approach. However, the final form of the problem is independent of the method used.

The boundary condition (5a) relates to the convective heat exchange at the unfinned side of the fin assembly, while the boundary condition (5d) represents the combined convective and radiative heat dissipation from the fin tip. The boundary conditions (5b) and (5c) arise from the assumption of perfect wall-to-fin contact which requires that the temperature and heat flux be continuous across the contact interface.

The mathematical complexity of the problem described by Eqs. (1-5) precludes a completely analytical treatment. However, closer inspection of these equations indicates that a certain amount of analytical manipulation is possible. The differential Eq. (1) can be integrated exactly to give

$$\varphi_w(X) = \alpha X + \beta \tag{6}$$

where α and β are, as yet, undetermined constants. This relation can be restated as

$$\varphi_w(X) = \frac{-\varphi_a X + \varphi_b(W+X)}{W} \tag{7}$$

where φ_a and φ_b denote the unknown surface temperatures of the wall; $\varphi_a = \varphi_w(X=-W)$ and $\varphi_b = \varphi_w(X=0)$. Combining Eq. (7) with the boundary condition (5a) gives rise to a relation between φ_a and φ_b and, accordingly, Eq. (7) becomes

$$\varphi_w(X) = \frac{-Bi_1 X + \varphi_b(Bi_1(X+(1+Bi_1 W)))}{(1+Bi_1 W)} \tag{8}$$

Thus, it only remains to determine the fin temperature distribution φ_f. By virtue of the perfect contact condition (5b) the fin-base temperature is identically equal to φ_b and, consequently, once the fin temperature distribution φ_f is determined then the wall temperature distribution φ_w can be simply computed from relation (8).

In order to determine the fin temperature distribution a modified implementation of the finite difference method[17] is devised. The modification is necessary in order to incorporate the radiation viewfactors which are defined explicitly only for the radiant interaction between isothermal surfaces.[18] The fin length is subdivided into $N-1$ equal sized elements of length H and nodes are situated at the endpoints of each of these elements, as shown schematically in Fig. 2. Assuming that the surface from $x_i - H/2$ to $x_i + H/2$ has constant temperature φ_i and uniform irradiation G_i (with appropriate modifications for x_1 and x_N), and then discretizing Eq. (2) using central difference approximations for the derivatives gives

$$\varphi_{i-1} - \left(2 + H^2 \frac{Bi_2}{\kappa T}\right)\varphi_i + \varphi_{i+1} - H^2 \epsilon_f \frac{Nc_2}{\kappa T}(\varphi_i^4 - G_i)$$
$$= -H^2 \frac{Bi_2}{\kappa T}\varphi_{2f} \tag{9}$$

Collocating expression (9) at each of the nodal points $x_1, x_2, \ldots, x_N$ generates a system of N nonlinear algebraic equations involving $2N+2$ unknowns; namely, $\varphi_0, \varphi_1, \ldots, \varphi_{N+1}$ and $G_1, G_2, \ldots, G_N$. An additional N equations

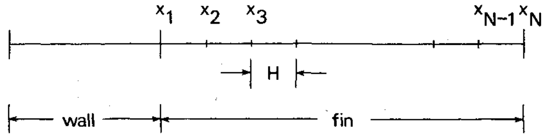

Fig. 2 Finite difference discretization of fin.

are generated by substituting for $G_w(0)$ from Eq. (3) into Eq. (4) and then collocating the discretized form of Eq. (4) at each of the nodal points $x_1, x_2, \ldots, x_N$. Thus, the combined algebraic representation constitutes $2N$ equations in $2N+2$ unknowns.

The number of unknowns is reduced to $2N$ by enforcing the boundary conditions (5c) and (5d). In boundary condition (5c), prior to discretization, all occurrences of $\varphi_w(x)$ are replaced appropriately using the expression (8), and $G_w(0)$ is replaced using relation (3). Then, discretizing with central difference approximations for the derivative gives rise to an explicit expression for φ_0 in terms of $\varphi_1, \varphi_2, \ldots, \varphi_N$ and $G_1, G_2, \ldots, G_N$. Similarly, discretizing boundary condition (5d) gives an explicit expression for φ_{n+1} in terms of φ_{n-1}. Thus, φ_0 and φ_{n+1} can be eliminated from the algebraic representation leaving a system of $2N$ nonlinear algebraic equations involving $2N$ unknowns; namely, $\varphi_1, \varphi_2, \ldots, \varphi_N$ and $G_1, G_2, \ldots, G_N$. The solution to this system of nonlinear algebraic equations is determined using the Newton-Raphson method[19] which is particularly appropriate for this problem as the differentiations for the elements of the Jacobian matrix[19] can be performed explicitly.

Two-Dimensional Formulation

For steady-state two-dimensional conductive heat flow, the inherent thermal symmetry indicates that it is only necessary to examine that section of the fin assembly bounded by the contour ABCDEFGA, Fig. 1. Thus, the determination of the fin assembly temperature distribution requires the simultaneous solution of

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right]\varphi_W(X,Y) = 0 \tag{10}$$

within the wall, and

$$\left[\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right]\varphi_f(X,Y) = 0 \tag{11}$$

within the fin, subject to the boundary conditions

$$\frac{\partial}{\partial X}\varphi_w(X,Y) = -Bi_1[1-\varphi_w(X,Y)] \quad \text{on AB} \tag{12a}$$

$$\frac{\partial}{\partial Y}\varphi_w(X,Y) = 0 \quad \text{on BC} \tag{12b}$$

$$\frac{\partial}{\partial Y}\varphi_w(X,Y) = 0 \quad \text{on CD} \tag{12c}$$

$$\frac{\partial}{\partial X}\varphi_f(X,Y) = -\frac{Bi_2}{\kappa}[\varphi_f(X,Y) - \varphi_{2f}]$$
$$-\epsilon_f \frac{Nc_2}{\kappa}[\varphi_f^4(X,Y) - \varphi_{2e}^4] \quad \text{on DE} \tag{12d}$$

$$\frac{\partial}{\partial Y}\varphi_f(X,Y) = -\frac{Bi_2}{\kappa}[\varphi_f(X,Y) - \varphi_{2f}]$$
$$-\epsilon_f \frac{Nc_2}{\kappa}[\varphi_f^4(X,Y) - G_f(X)] \quad \text{on EF} \tag{12e}$$

$$\varphi_w(X,Y) = \varphi_f(X,Y) \quad \text{on FC} \tag{12f}$$

$$\frac{\partial}{\partial X}\varphi_w(X,Y)=\kappa\frac{\partial}{\partial X}\varphi_f(X,Y) \quad \text{on FC} \tag{12g}$$

$$\frac{\partial}{\partial X}\varphi_w(X,Y)=-Bi_2[\varphi_w(X,Y)-\varphi_{2f}]$$

$$-\epsilon_w Nc_2[\varphi_w^4(X,Y)-G_w(Y)] \quad \text{on FG} \tag{12h}$$

$$\frac{\partial}{\partial X}\varphi_w(X,Y)=0 \quad \text{on GA} \tag{12i}$$

combined with the irradiation equations

$$G_w(Y)=\int_{X=0}^{L}F_{dY-dX}[\epsilon_f\varphi_f^4(X,Y)+(1-\epsilon_f)G_f(X)]$$

$$+\int_{X'=0}^{L}F_{dY-dX'}(\epsilon_f\varphi_f^4(X,Y)+(1-\epsilon_f)G_f(X))$$

$$+F_{dY-EE'}\varphi_{2e}^4 \tag{13}$$

and

$$G_f(X)=\int_{Y=T}^{P}F_{dX-dY}(\epsilon_w\varphi_w^4(X,Y)+(1-\epsilon_w)G_w(Y))$$

$$+\int_{Y'=T}^{P}F_{dX-dY'}(\epsilon_w\varphi_w^4(X,Y)+(1-\epsilon_w)G_w(Y))$$

$$+F_{dX-EE'}\varphi_{2e}^4 \tag{14}$$

The boundary conditions (12b), (12c), and (12i) arise from the geometrical symmetry of the fin arrangement and stipulate that there is no heat flux across the fictitious boundaries BC, CD, and GA. The boundary conditions (12f) and (12g) result from the assumption of perfect wall-to-fin contact which requires that the temperature and heat flux be continuous across the interface FC. The remaining boundary conditions describe the heat exchange at the exposed surfaces AB and DEFG.

The irradiation Eqs. (13) and (14) are derived by considering the radiant flux within the enclosure EFF′E′E, and include the effects of fin-to-environment, fin-to-fin, fin-to-wall, and wall-to-environment radiant interactions. These equations are more complicated than the corresponding relations for the one-dimensional model, namely, Eqs. (3) and (4). This is due to the fact that in the two-dimensional model both the temperature and surface irradiation vary along the base-surface FG, while in the one-dimensional model this surface behaves as an isothermal surface with uniform irradiation.

The solution of the problem described by Eqs. (10-14) is susceptible to treatment by various numerical techniques, e.g., the finite difference,[17] finite element,[20] and boundary integral equation[21] (BIE) methods. In this study a recently developed nonlinear implementation of the BIE method[16] is used because it is computationally more efficient than the other two methods, and has the inherent advantage that it is directly compatible with the irradiation relations, Eqs. (13) and (14).

With the BIE method, the problem described by Eqs. (10-14) is first transformed into a pair of coupled integral equations involving contour integrals around ABCFGA and CDEFC[16,21]; the coupling arises as a consequence of the interface boundary conditions (12f) and (12g). In order to effect a solution to these integral equations, the contours ABCFGA and CDEFC are subdivided into a total of N segments and nodes are situated at the midpoint of each of these segments. The temperature, heat flux, and, where appropriate, surface irradiation on each segment are approximated by piecewise-constant functions. Then, the discretized form of the integral equations is collocated at each of the boundary nodes. This generates a system of N nonlinear algebraic equations involving the N unknown nodal temperatures and, in addition, N(EF) + N(FG) unknown values of the surface irradiation from the segments on the surfaces EF and FG. An additional N(EF) + N(FG) nonlinear algebraic equations are generated by discretizing the irradiation relations (13) and (14). The solution to the combined system of N + N(EF) + N(FG) nonlinear algebraic equations in N + N(EF) + N(FG) unknowns is achieved readily employing the Newton-Raphson method[19] combined with a Gaussian elimination scheme[19] to solve the linearized equations.

Heat Exchanger Performance

The heat flow rate through the fin assembly is expressed most conveniently in the form of an augmentation factor defined as the ratio of the heat flow rate of the fin assembly to that of the unfinned wall operating under the same conditions. Thus, the augmentation factor is a measure of the improvement in the overall heat flow rate resulting from the addition of fins. Therefore, it is more useful for design purposes than the widely used fin efficiency (e.g., Refs. 1-3) which only gives an indication of the fin heat dissipation. Furthermore, it differs from the conventional fin effectiveness[1] in that the latter fails to account for either the conductive heat flow within the supporting surface or the convective heat exchange at the plain side of the supporting surface.

In order to evaluate the augmentation factor it is first necessary to determine the heat flow rate through the unfinned wall. This requires the solution of the energy equation

$$\frac{d^2}{dX^2}\varphi_w^*(X)=0 \tag{15}$$

within the wall, subject to the boundary conditions

$$\frac{d}{dX}\varphi_w^*(X)=-Bi_1[1-\varphi_w^*(X)] \quad \text{at } X=-W \tag{16a}$$

$$\frac{d}{dX}\varphi_w^*(X)=-Bi_2[\varphi_w^*(X)-\varphi_{2f}]-\epsilon_w Nc_2[\varphi_w^{*4}(X)-\varphi_{2e}^4]$$

$$\text{at } X=0 \tag{16b}$$

where the energy equation (15) is obtained by performing an energy balance on an infinitesimal element of the wall, and the boundary conditions (16) describe the heat exchange at the surfaces of the wall.

It is not possible to derive a closed-form solution for the problem defined by Eqs. (15) and (16) because of the nonlinearities occurring in the boundary condition (16b). However, the need to use numerical techniques, such as the finite difference method, can be avoided by some simple analytical manipulation. Integrating Eq. (15) exactly, and then enforcing the boundary condition (16a) gives

$$\varphi_w^*(X)=-\frac{Bi_1X+\varphi_b^*(Bi_1X+(1+Bi_1W))}{(1+Bi_1W)} \tag{17}$$

where φ_b^* denotes the unknown temperature of the surface $x=0$. Combining the relation (17) with the boundary condition (16b) gives a nonlinear algebraic equation involving φ_b^*

$$\left(1+Bi_2 W-\frac{1}{1+Bi_1 W}\right)\varphi_b^*+\epsilon_w Nc_2 W\varphi_b^{*4} = \frac{Bi_1 W}{1+Bi_1 W}+Bi_2 W\varphi_{2f}+\epsilon_w Nc_2 W\varphi_{2e}^4 \tag{18}$$

The solution to Eq. (18) is achieved readily employing the Newton-Raphson method.[19] This determines φ_b^* and then the temperature distribution within the wall, $\varphi_w^*(X)$, can be computed employing the relation (17). The corresponding heat flow rate is given by

$$Q^*=-k_w\frac{\partial}{\partial X}\varphi_w^*(X=-W)\theta_1=k_w\frac{Bi_1}{1+Bi_1 W}(1-\varphi_b^*)\theta_1 \tag{19}$$

and, therefore, using the appropriate one- and two-dimensional solutions

$$\text{AUG1}=Q1/Q^*=\frac{1-\varphi_b}{1-\varphi_b^*} \tag{20}$$

and

$$\text{AUG2}=Q2/Q^*=-\frac{1+Bi_1 W}{1-\varphi_b^*}\int_{AB}[1-\varphi_w(X,Y)]\,\mathrm{d}S \tag{21}$$

where dS denotes an incremental element of AB.

Accuracy of Numerical Calculations

The finite difference method employed in conjunction with the one-dimensional formulation, and the BIE method used with the two-dimensional formulation both involve the introduction of approximations in order to determine the fin assembly temperature distribution. Consequently, the corresponding solutions invariably include errors. These errors are related to the associated mesh sizes[17,21] and, therefore, diminish as the respective discretization is refined. In order to check for this convergence, for each particular problem examined, solutions were computed for three different levels of discretization corresponding to a uniform refinement of the appropriate mesh in the ratio of 1:½:¼. The one-dimensional finite difference solutions were computed by subdividing the fin length into 25, 50, and 100 elements, while the two-dimensional BIE solutions were obtained by subdividing the contours ABCFGA and CDEFC into a total of 80, 160, and 320 segments. The distribution of the BIE boundary segments accounted for the fact that, in practice, the fin length is by far the largest dimension of the section ABCDEFGA and, consequently, the sides CD and EF were subdivided into four times as many segments as the sides AX, XB, BC, CF, FG, GA, DE, and FC. This particular discretization was found to offer the most efficient use of the available computational resources.

Solutions were computed for a large number of problems. It was found that both the one- and two-dimensional solutions display a monotonic convergent behavior as the respective discretization is refined. The one-dimensional solutions converge quite rapidly; the 50 and 100 element solutions always agree to at least four significant figures. In contrast, the 160 and 320 segment BIE solutions only agree to two significant figures. As further refinement of the boundary discretization is impractical, the limiting values of AUG2 were computed by extrapolation employing Richardson's formula[19]

$$\text{ERR}(N)\,\alpha\,(H(N))^{\alpha} \tag{22}$$

where ERR(N) is the error in the N segment solution, $H(N)$ is an associated segment length and α is the order of the extrapolation.

Results and Discussion

For both the one- and two-dimensional formulations a complete description of the fin assembly heat flow requires the specification of the eleven independent parameters: Bi_1, Bi_2, κ, L, T, W, Nc_2, ϵ_w, ϵ_f, φ_{2e}, and φ_{2f}. The excessive computational time necessary to determine solutions for a complete range of these parameters precludes the possibility of performing a conventional parametric study. Therefore, we adopted a more practical procedure. For each particular problem considered, the parameters κ, L, T, W, ϵ_w, and ϵ_f were assigned prescribed values, and then solutions were computed for a comprehensive range of the parameters Bi_1, Bi_2, Nc_2, φ_{2e}, and φ_{2f}. Essentially, this is equivalent to in-

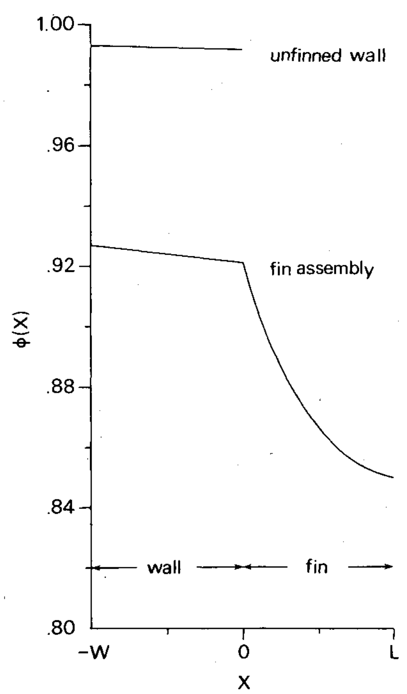

Fig. 3 One-dimensional solutions for problem A.

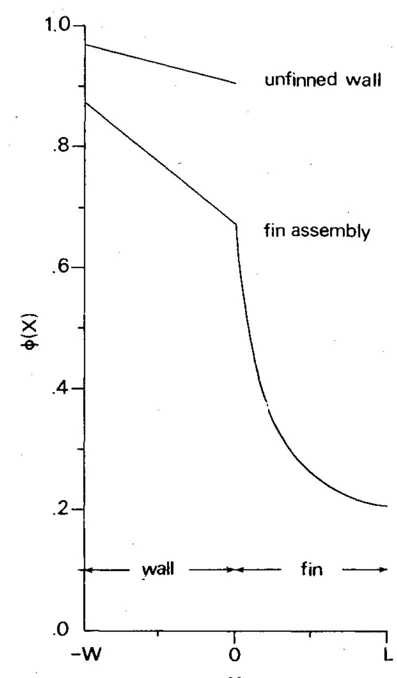

Fig. 4 One-dimensional solutions for problem B.

Table 1 System parameters

Problem	Bi_1	Bi_2	κ	L	T	W	Nc_2	ϵ_w	ϵ_f	φ_{2e}	φ_{2f}
A	2×10^{-2}	10^{-3}	10	20	0.2	0.4	10^{-3}	0.6	0.8	0.1	0.2
B	4.0	2×10^{-1}	20	20	0.2	0.4	10^{-3}	0.6	0.9	0.1	0.2

vestigating the performance of a given heat exchanger under different operating conditions. The most significant results arising from this investigation are discussed in detail in the following.

Errors in the Conventional Analyses

In all previous published work on the subject of radiant heat dissipation from finned surfaces, there is the inherent assumption that the temperature of the surface to which the fins are attached is unaffected by the addition of the fins. An investigation has been performed in order to determine the applicability of this assumption. This investigation involved the comparison of the solutions for the unfinned wall and the fin assembly. The results for two particular problems are presented in Figs. 3 and 4, which show the temperature distributions within the unfinned wall and the fin assembly for the problems defined by the system parameters as given in Table 1. Problem A is representative of a heat exchanger comprised of a stainless-steel wall with aluminum fins operating with forced convection of air on the plain side and combined free convection and radiation on the fin side. Problem B describes the performance of a heat exchanger consisting of a stainless-steel wall with copper fins subject to forced convection of water on the plain side and combined forced convection of air and radiation on the fin side.

The results for these two problems are characteristic of those observed for other values of the system parameters and illustrate, in particular, that the addition of fins acts to reduce the temperature of the base surface. This reduction is 7% for problem A, and 25% for problem B. In fact, it has been found that the reduction can be as much as 50%. Thus, the assumption that the temperature of the base surface is unaffected by the addition of fins is clearly inappropriate.

In order to determine the extent of the inaccuracies in the heat flow rates predicted by the isothermal base-surface analyses, solutions have been computed assuming that the temperature distribution within the wall remains as predicted by the unfinned wall analysis. These solutions were obtained by a relatively simple algebraic manipulation of the fin assembly solutions. The temperature distribution within the fin, for the isothermal base-surface situation, is given by

$$\hat{\varphi}_f(X) = \varphi_f(X)\,(\varphi_b^*/\varphi_b) \tag{23}$$

where $\varphi_f(X)$ is the fin temperature distribution given by the fin assembly formulation, and φ_b^* and φ_b are the temperatures of the base surface in the unfinned and finned states, respectively. The corresponding augmentation factor is given by

$$\begin{aligned}\hat{AUG}1 = \frac{(1+Bi_1)}{Bi_1(1-\varphi_b^*)}\{Bi_2(1-T)(\hat{\varphi}_f(0)-\varphi_{2f})\} \\ +\epsilon_w Nc_2(1-T)[\hat{\varphi}_f^4(0)-G_w(0)] \\ +\int_{AB}\{Bi_2[\hat{\varphi}_f(X)-\varphi_{2f}]+\epsilon_f Nc_2[\hat{\varphi}_f^4(X)-G_f(X)]\}dS \\ +Bi_2T[\hat{\varphi}_f(L)-\varphi_{2f}]+\epsilon_f Nc_2 T[\hat{\varphi}_f^4(L)-\varphi_{2e}^4]\end{aligned} \tag{24}$$

where dS denotes an incremental element of AB.

A comparison of the augmentation factors AÛG1 and AUG1 has been performed for an extensive range of problems. The results for problems A and B are given in Table 2. These results show that the error in AÛG1 is in excess of 35% for problem A, and in excess of 90% for problem B. In fact, it has been found that in some cases these errors can exceed 150%. It is, therefore, apparent that the isothermal base-surface analyses are grossly inappropriate for the effective design of finned heat exchangers.

Table 2 Results for problems A and B

Problem	AÛG1	AUG1	AÛG1/AUG1, %
A	14.48	10.68	135
B	6.81	3.53	192

Errors in the One-Dimensional Analysis

As even the most advanced of the previously employed analyses for examining convecting and radiating finned surfaces have been based upon the isothermal base-surface assumption, it may be concluded that the one-dimensional fin assembly formulation presented in this paper will facilitate an improvement in the design of finned heat exchangers. However, even this formulation may involve appreciable errors since it is based upon a unidirectional heat flow analysis; several recent investigations (e.g., Refs. 7-9), considering finned surfaces involving purely convective surface heat exchange, have shown that the one-dimensional

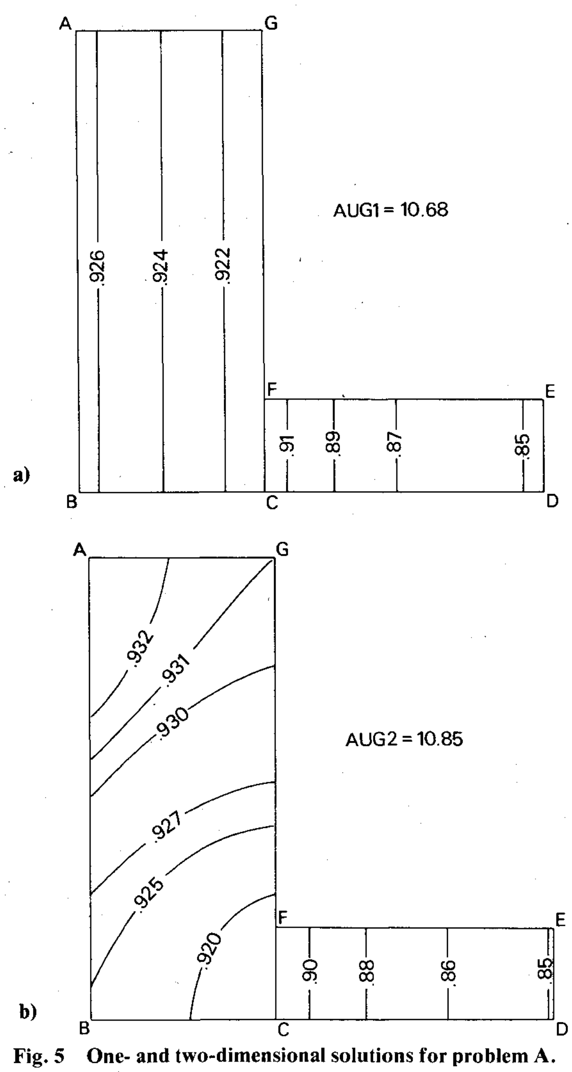

Fig. 5 One- and two-dimensional solutions for problem A.

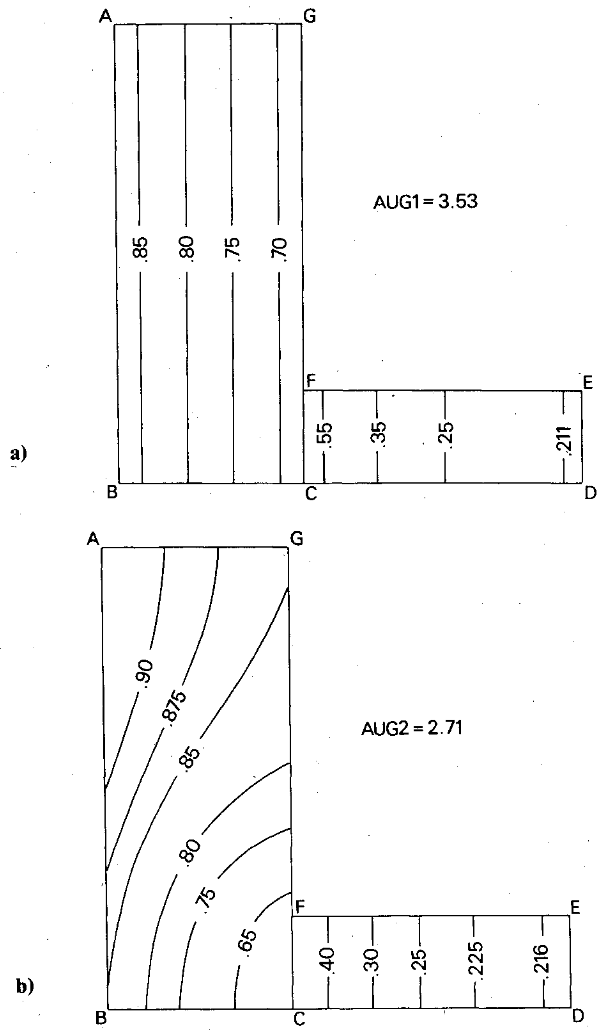

Fig. 6 One- and two-dimensional solutions for problem B.

approximation can result in errors of up to 80% in the prediction of the heat flow rate. Therefore, we performed an investigation in order to determine the validity of the one-dimensional approximation for problems involving combined convective and radiative surface heat exchange. The one- and two-dimensional solutions were compared for a large number of problems, and the solutions to problems A and B are displayed in Figs. 5 and 6. These solutions illustrate some of the deficiencies in the one-dimensional analysis and are consistent with the effects evident in other problems. The most significant feature of the two-dimensional solutions is the temperature depression at the fin-base. This temperature depression has been observed in the previous two-dimensional investigation[7-9] and is attributed to the fact that a greater proportion of the heat flow is channelled through the fin base than through the adjacent unfinned portion of the base surface. However, this temperature depression is not accounted for in the one-dimensional analysis which considers the temperature at the base surface to be identical to that at the adjacent unfinned portion of the base surface. This deficiency of the one-dimensional model has been found to result in an underestimation of the overall heat flow rate by 2% for problem A, and an overestimation of the heat flow rate by 27% for problem B. In fact, errors of over 30% have been observed for problems in the range $Bi_l < 10^{-1}$ and $Bi_2 < 10^{-2}$. Thus, it is apparent that a multidirectional analysis is essential in order to achieve an accurate prediction of the performance of finned heat exchangers.

Conclusions

One- and two-dimensional formulations have been developed for analyzing the heat flow within finned surfaces which operate under conditions involving both convective and radiative surface heat exchange. These formulations take into account the heat conduction within the supporting surface and the surface heat exchange at the plain side of the supporting surface. A most significant feature of the results predicted by these formulations is the reduction in the temperature of the surface to which the fins are attached. Since all previous investigations have been based upon the assumption that the temperature of this surface is unaffected by the presence of the fins, it is apparent that the work presented in this paper will facilitate an improvement in the design of finned heat exchangers.

A comparison of the one- and two-dimensional solutions shows that errors of up to 30% can occur in the one-dimensional model. This is a consequence of the fact that the one-dimensional analysis cannot model accurately the temperature depression which occurs at the fin base. Thus, the effective design of finned heat exchangers requires a two-dimensional analysis which accounts for the thermal interaction between the fins and the supporting surface.

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